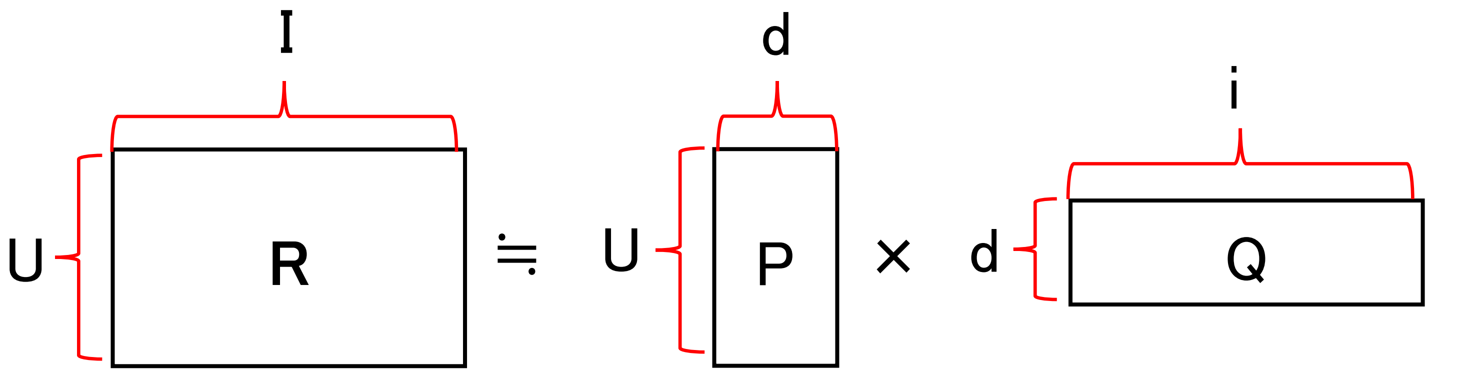
無回答の商品は購入していないと仮定して、レコメンドの対象とする
↓
評価のそこそこ高い商品をレコメンドする
↓
欠測（O）の評価値を推測

③行列因子分解

評価値を分解した潜在因子行列を解析的に求めて予測値を算出

d： 潜在因子（価格、カテゴリ、隠れた特性など）
Rが4×4であれば、4×1と1×4、4×3と3×4あるいは4×2と2×4の組合せでも構いません



	item1	item2	item3	item4	⋯
user1	2	3	−	5	⋯
user2	2	5	−	5	⋯
user3	−	3	4	4	⋯
user4	4	2	3	−	⋯
⋮	⋮	⋮	⋮	⋮	⋱

ユーザー因子行列　アイテム因子行列

潜在因子行列

実際の評価値

$$R = \begin{pmatrix} 2 & 3 & 0 & 5 \\ 2 & 5 & 0 & 5 \\ 0 & 3 & 4 & 4 \\ 4 & 2 & 3 & 0 \end{pmatrix}$$

一般化すると

$$R = \begin{pmatrix} r_{1,1} & r_{1,2} & r_{1,3} & r_{1,4} \\ r_{2,1} & r_{2,2} & r_{2,3} & r_{2,4} \\ r_{3,1} & r_{3,2} & r_{3,3} & r_{3,4} \\ r_{4,1} & r_{4,2} & r_{4,3} & r_{4,4} \end{pmatrix}$$

潜在因子行列

$\hat{\mathrm{R}} = P\ Q$ のように近似的に分解できるとする

ユーザー因子行列

$$P = \begin{pmatrix} p_{1,1} & p_{1,2} \\ p_{2,1} & p_{2,2} \\ p_{3,1} & p_{3,2} \\ p_{4,1} & p_{4,2} \end{pmatrix}$$

アイテム因子行列

$$Q = \begin{pmatrix} q_{1,1} & q_{1,2} & q_{1,3} & q_{1,4} \\ q_{2,1} & q_{2,2} & q_{2,3} & q_{2,4} \end{pmatrix}$$

$$R = \begin{pmatrix} r_{1,1} & r_{1,2} & r_{1,3} & r_{1,4} \\ r_{2,1} & r_{2,2} & r_{2,3} & r_{2,4} \\ r_{3,1} & r_{3,2} & r_{3,3} & r_{3,4} \\ r_{4,1} & r_{4,2} & r_{4,3} & r_{4,4} \end{pmatrix} \fallingdotseq \begin{pmatrix} p_{1,1} & p_{1,2} \\ p_{2,1} & p_{2,2} \\ p_{3,1} & p_{3,2} \\ p_{4,1} & p_{4,2} \end{pmatrix} \begin{pmatrix} q_{1,1} & q_{1,2} & q_{1,3} & q_{1,4} \\ q_{2,1} & q_{2,2} & q_{2,3} & q_{2,4} \end{pmatrix} = \hat{\mathrm{R}}$$

←近似値なので^（ハットを添字）

$$= \begin{pmatrix} p_{1,1}q_{1,1} + p_{1,2}q_{2,1} & p_{1,1}q_{1,2} + p_{1,2}q_{2,2} & p_{1,1}q_{1,3} + p_{1,2}q_{2,3} & p_{1,1}q_{1,4} + p_{1,2}q_{2,4} \\ p_{2,1}q_{1,1} + p_{2,2}q_{2,1} & p_{2,1}q_{1,2} + p_{2,2}q_{2,2} & p_{2,1}q_{1,3} + p_{2,2}q_{2,3} & p_{2,1}q_{1,4} + p_{2,2}q_{2,4} \\ p_{3,1}q_{1,1} + p_{3,2}q_{2,1} & p_{3,1}q_{1,2} + p_{3,2}q_{2,2} & p_{3,1}q_{1,3} + p_{3,2}q_{2,3} & p_{3,1}q_{1,4} + p_{3,2}q_{2,4} \\ p_{4,1}q_{1,1} + p_{4,2}q_{2,1} & p_{4,1}q_{1,2} + p_{4,2}q_{2,2} & p_{4,1}q_{1,3} + p_{4,2}q_{2,3} & p_{4,1}q_{1,4} + p_{4,2}q_{2,4} \end{pmatrix}$$

$r_{1,1} \fallingdotseq p_{1,1}q_{1,1} + p_{1,2}q_{2,1}$

$r_{1,1} - (p_{1,1}q_{1,1} + p_{1,2}q_{2,1}) \fallingdotseq 0$ ←近似値が実際の評価値と同じ場合は0になる

$$R - \hat{\mathrm{R}} = R - PQ = \begin{pmatrix} r_{1,1} - (p_{1,1}q_{1,1} + p_{1,2}q_{2,1}) & \cdots & \cdots & r_{1,1} - (p_{1,1}q_{1,1} + p_{1,2}q_{2,1}) \\ \vdots & \cdots & & \vdots \\ & \cdots & & \\ r_{1,1} - (p_{1,1}q_{1,1} + p_{1,2}q_{2,1}) & \cdots & \cdots & r_{1,1} - (p_{1,1}q_{1,1} + p_{1,2}q_{2,1}) \end{pmatrix} = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & \cdots & \cdots & \vdots \\ \vdots & \cdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}$$

Rの近似値を$\hat{R}$とおきます

$R-\hat{R}=E$　誤差（残差）

$$R-\hat{R}=\begin{pmatrix} r_{1,1} & r_{1,2} & r_{1,3} & r_{1,4} \\ r_{2,1} & r_{2,2} & r_{2,3} & r_{2,4} \\ r_{3,1} & r_{3,2} & r_{3,3} & r_{3,4} \\ r_{4,1} & r_{4,2} & r_{4,3} & r_{4,4} \end{pmatrix} - \begin{pmatrix} p_{1,1}q_{1,1}+p_{1,2}q_{2,1} & p_{1,1}q_{1,2}+p_{1,2}q_{2,2} & p_{1,1}q_{1,3}+p_{1,2}q_{2,3} & p_{1,1}q_{1,4}+p_{1,2}q_{2,4} \\ p_{2,1}q_{1,1}+p_{2,2}q_{2,1} & p_{2,1}q_{1,2}+p_{2,2}q_{2,2} & p_{2,1}q_{1,3}+p_{2,2}q_{2,3} & p_{2,1}q_{1,4}+p_{2,2}q_{2,4} \\ p_{3,1}q_{1,1}+p_{3,2}q_{2,1} & p_{3,1}q_{1,2}+p_{3,2}q_{2,2} & p_{3,1}q_{1,3}+p_{3,2}q_{2,3} & p_{3,1}q_{1,4}+p_{3,2}q_{2,4} \\ p_{4,1}q_{1,1}+p_{4,2}q_{2,1} & p_{4,1}q_{1,2}+p_{4,2}q_{2,2} & p_{4,1}q_{1,3}+p_{4,2}q_{2,3} & p_{4,1}q_{1,4}+p_{4,2}q_{2,4} \end{pmatrix}$$

$$=\begin{pmatrix} r_{1,1}-(p_{1,1}q_{1,1}+p_{1,2}q_{2,1}) & \cdots & \cdots & r_{1,4}-(p_{1,1}q_{1,4}+p_{1,2}q_{2,4}) \\ \vdots & \cdots & & \vdots \\ & \cdots & & \\ r_{4,1}-(p_{4,1}q_{1,1}+p_{4,2}q_{2,1}) & \cdots & \cdots & r_{4,4}-(p_{4,1}q_{1,4}+p_{4,2}q_{2,4}) \end{pmatrix} = E$$ 残差行列を最小化にする

L:　損失関数

$$L=r_{1,1}-(p_{1,1}q_{1,1}+p_{1,2}q_{2,1})+\cdots+r_{1,4}-(p_{1,1}q_{1,4}+p_{1,2}q_{2,4})+\cdots+r_{4,4}-(p_{4,1}q_{1,4}+p_{4,2}q_{2,4})$$
$$=\sum_{u=1}^{4}\sum_{i=1}^{4}\{r_{u,i}-(p_{u,1}q_{1,i}+p_{u,2}q_{2,u})\}=\sum_{u=1}^{4}\sum_{i=1}^{4}\{r_{u,i}-(\sum_{d=1}^{2}p_{u,d}q_{d,i})\}$$

$$L=\sum_{u-1}^{4}\sum_{i=1}^{4}\{r_{u,i}-(\sum_{d=1}^{2}p_{u,d}q_{d,i})\}^2$$

$$l=\{r_{u,i}-(\sum_{d=1}^{2}p_{u,d}q_{d,i})\}^2=(r_{u,i})^2-2r_{u,i}(\sum_{d=1}^{2}p_{u,d}q_{d,i})+(\sum_{d=1}^{2}p_{u,d}q_{d,i})^2$$
$$=(r_{u,i})^2-2r_{u,i}\,p_{u,1}q_{1,i}-2r_{u,i}\,p_{u,2}q_{2,i}+{p_{u,1}}^2{q_{1,i}}^2+2p_{u,1}q_{1,i}p_{u,2}q_{2,i}+{p_{u,2}}^2\,{q_{2,i}}^2$$

４つの未知数$p_{u,1}$、$p_{u,2}$、 $q_{1,i}$ 、$q_{2,i}$で偏微分する

$$l = (r_{u,i})^2 - 2r_{u,i}\, p_{u,1}q_{1,i} - 2r_{u,i}\, p_{u,2}q_{2,i} + {p_{u,1}}^2 {q_{1,i}}^2 + 2p_{u,1}q_{1,i}p_{u,2}q_{2,i} + {p_{u,2}}^2\, {q_{2,i}}^2$$

$p_{u,1}$で偏微分する $$\frac{\partial l}{\partial p_{u,1}} = -2\{2r_{u,i} - (p_{u,1}q_{1,i} + p_{u,2}q_{2,i})\}\, q_{1,i} = -2e_{u,i}\, q_{1,i}$$

$p_{u,2}$で偏微分する $$\frac{\partial l}{\partial p_{u,2}} = -2\{2r_{u,i} - (p_{u,1}q_{1,i} + p_{u,2}q_{2,i})\}q_{2,i} = -2e_{u,i}\, q_{2,i}$$

$q_{1,i}$で偏微分する $$\frac{\partial l}{\partial q_{1,i}} = -2\{2r_{u,i} - (p_{u,1}q_{1,i} + p_{u,2}q_{2,i})\}p_{u,1} = -2e_{u,i}\, p_{u,1}$$

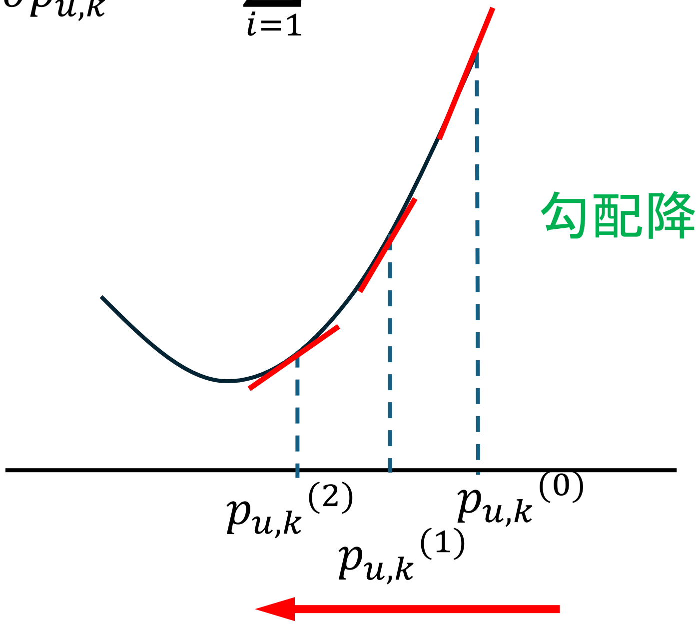
$q_{2,i}$で偏微分する $$\frac{\partial l}{\partial q_{2,i}} = -2\{2r_{u,i} - (p_{u,1}q_{1,i} + p_{u,2}q_{2,i})\}p_{u,2} = -2e_{u,i}\, p_{u,2}$$

$$e_{u,i} = 2r_{u,i} - (p_{u,1}q_{1,i} + p_{u,2}q_{2,i})$$

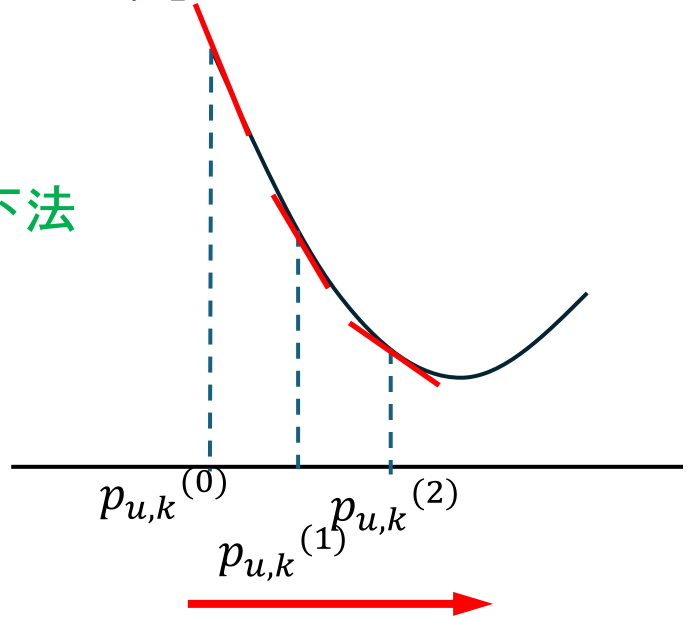
$$R - \hat{R} = E = \begin{pmatrix} e_{1,1} & e_{1,2} & e_{1,3} & e_{1,4} \\ e_{2,1} & e_{2,2} & e_{2,3} & e_{2,4} \\ e_{3,1} & e_{3,2} & e_{3,3} & e_{3,4} \\ e_{4,1} & e_{4,2} & e_{4,3} & e_{4,4} \end{pmatrix}$$

$$\frac{\partial L}{\partial p_{u,k}} = -2\sum_{i=1}^{4} e_{u,i}\, q_{k,i}$$

$$\frac{\partial L}{\partial q_{k,i}} = -2\sum_{i=1}^{4} e_{u,i}\, p_{u,,k}$$

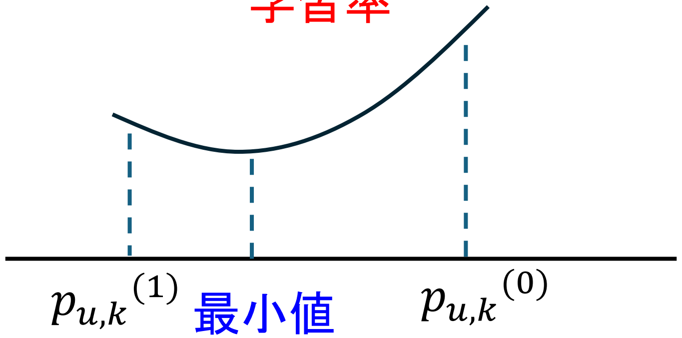


勾配降下法



$$\begin{aligned} {p_{u,k}}^{(1)} &= {p_{u,k}}^{(0)} - \eta \frac{\partial L}{{p_{u,k}}^{(0)}} \\ &= {p_{u,k}}^{(0)} - \eta\left(-2\textstyle\sum_{i=1}^{4} e_{u,i}\, {q_{k,i}}^{(0)}\right) \\ &= {p_{u,k}}^{(0)} + 2\eta \textstyle\sum_{i=1}^{4} e_{u,i}\, {q_{k,i}}^{(0)} \end{aligned}$$

学習率



$$R=\begin{pmatrix}2&3&\color{red}{0}&5\\2&5&\color{red}{0}&5\\\color{red}{0}&3&4&4\\4&2&3&\color{red}{0}\end{pmatrix}\qquad P=\begin{pmatrix}0.37&0.95\\0.73&0.60\\0.16&0.16\\0.06&0.87\end{pmatrix}\qquad Q=\begin{pmatrix}0.60&0.71&0.02&0.97\\0.83&0.21&0.18&0.18\end{pmatrix}$$

$$\hat{R}=PQ=\begin{pmatrix}1.011&0.46&0.18&0.53\\0.94&0.64&0.12&0.82\\0.23&0.15&0.03&0.18\\0.76&0.23&0.16&0.21\end{pmatrix}$$

欠損部分は「－」

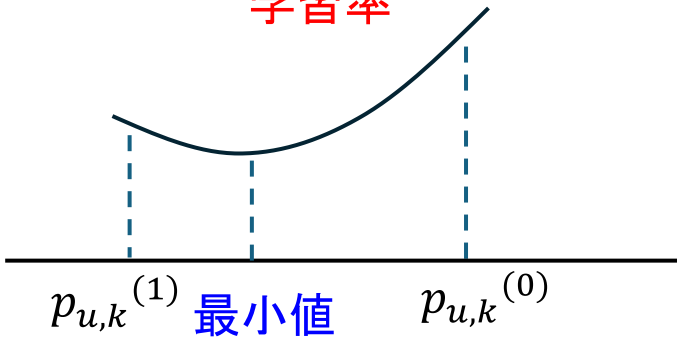
$$E=R-\hat{R}=\begin{pmatrix}2&3&\color{red}{0}&5\\2&5&\color{red}{0}&5\\\color{red}{0}&3&4&4\\4&2&3&\color{red}{0}\end{pmatrix}-\begin{pmatrix}1.011&0.46&0.18&0.53\\0.94&0.64&0.12&0.82\\0.23&0.15&0.03&0.18\\0.76&0.23&0.16&0.21\end{pmatrix}=\begin{pmatrix}0.99&2.54&\color{red}{-}&4.47\\1.06&4.36&\color{red}{-}&4.18\\\color{red}{-}&2.85&3.97&3.82\\3.24&1.77&2.84&\color{red}{-}\end{pmatrix}$$

$$L=\sum_{u=1}^{4}\sum_{i=1}^{4}\left\{r_{u,i}-\left(\sum_{d=1}^{2}p_{u,d}q_{d,i}\right)\right\}^2=\sum_{u=1}^{4}\sum_{i=1}^{4}e_{u,i}{}^2\fallingdotseq 125.10$$

$$\begin{aligned}p_{u,k}{}^{(1)}&=p_{u,k}{}^{(0)}-\color{red}{\eta}\frac{\partial L}{p_{u,k}{}^{(0)}}\\&=p_{u,k}{}^{(0)}-\color{red}{\eta}\left(-2\sum_{i=1}^{4}e_{u,i}\,q_{k,i}{}^{(0)}\right)\\&=p_{u,k}{}^{(0)}+2\color{red}{\eta}\sum_{i=1}^{4}e_{u,i}\,q_{k,i}{}^{(0)}\end{aligned}$$

学習率

$$\begin{aligned}p_{1,1}{}^{(1)}&=p_{1,1}{}^{(0)}+2\color{red}{\eta}\sum_{i=1}^{4}e_{1,i}\,q_{1,i}{}^{(0)}\\&=0.37+2\times0.01\{0.99\times0.60+2.54\times0.71+4.47\times0.97\}\fallingdotseq 0.51\\q_{1,1}{}^{(1)}&=q_{1,1}{}^{(0)}+2\color{red}{\eta}\sum_{u=1}^{4}e_{u,1}\,p_{u,1}{}^{(0)}\\&=0.60+2\times0.01\{0.99\times0.37+1.06\times0.73+3.24\times0.06\}\fallingdotseq 0.63\end{aligned}$$



$$P=\begin{pmatrix}0.51&0.99\\0.89&0.66\\0.28&0.21\\0.13&0.94\end{pmatrix}\qquad Q=\begin{pmatrix}0.63&0.81&0.04&1.11\\0.92&0.36&0.25&0.34\end{pmatrix}\qquad PQ=\begin{pmatrix}1.034&0.77&0.27&0.90\\1.17&0.96&0.20&1.21\\0.37&0.31&0.07&0.30\\0.95&0.44&0.24&0.46\end{pmatrix}$$

2回目

$$E = R - \hat{R} = \begin{pmatrix} 0.76 & 2.23 & - & 4.10 \\ 0.83 & 4.04 & - & 3.79 \\ - & 2.69 & 3.93 & 3.61 \\ 3.05 & 1.56 & 2.76 & - \end{pmatrix}$$

$$L = \sum_{u=1}^{4} \sum_{i=1}^{4} {e_{u,i}}^2 \fallingdotseq 108.80$$

1回目

$$E = R - \hat{R} = \begin{pmatrix} 0.99 & 2.54 & - & 4.47 \\ 1.06 & 4.36 & - & 4.18 \\ - & 2.85 & 3.97 & 3.82 \\ 3.24 & 1.77 & 2.84 & - \end{pmatrix}$$

$$L = \sum_{u=1}^{4} \sum_{i=1}^{4} {e_{u,i}}^2 \fallingdotseq 108.80$$

10,000回施行

$$P = \begin{pmatrix} 1.25 & 1.19 \\ 2.36 & 0.84 \\ 1.33 & 0.85 \\ 0.29 & 3.09 \end{pmatrix} \quad Q = \begin{pmatrix} 0.40 & 1.96 & 2.54 & 1.01 \\ 1.26 & 0.46 & 0.73 & 3.14 \end{pmatrix} \quad PQ = \begin{pmatrix} 2.00 & 3.00 & 4.05 & 5.00 \\ 2.00 & 5.00 & 5.00 & 5.00 \\ 1.60 & 3.00 & 4.00 & 4.00 \\ 4.00 & 2.00 & 3.00 & 5.00 \end{pmatrix}$$

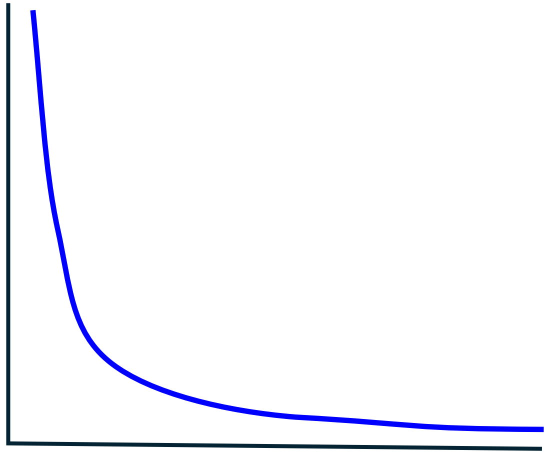
$$E = R - \hat{R} = \begin{pmatrix} 0 & 0 & - & 0 \\ 0 & 0 & - & 0 \\ - & 0 & 0 & 0 \\ 0 & 0 & 0 & - \end{pmatrix}$$

欠損値が算出

$$R = \begin{pmatrix} 2 & 3 & 0 & 5 \\ 2 & 5 & 0 & 5 \\ 0 & 3 & 4 & 4 \\ 4 & 2 & 3 & 0 \end{pmatrix}$$

$$L = \sum_{u=1}^{4} \sum_{i=1}^{4} {e_{u,i}}^2 = 0$$ ←損失ゼロ

残差行列Eの各要素の2乗和



試行回数

$$R = \begin{pmatrix} 2 & 3 & 0 & 5 \\ 2 & 5 & 0 & 5 \\ 0 & 3 & 4 & 4 \\ 4 & 2 & 3 & 0 \end{pmatrix}$$

協調フィルタリングによる予測例

	item1	item2	item3	item4
user 1	2	3	4	5
user 2	2	5	5	5
user 3	2	3	4	4
user 4	4	2	3	3

行列因子分解の実行例

$$PQ = \begin{pmatrix} 2.00 & 3.00 & 4.05 & 5.00 \\ 2.00 & 5.00 & 5.00 & 5.00 \\ 1.60 & 3.00 & 4.00 & 4.00 \\ 4.00 & 2.00 & 3.00 & 5.00 \end{pmatrix}$$