1列

1行 $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ ⇒

(1, 1)成分の余因子

$$(-1)^{1+1} \times \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

1行目で余因子展開すると

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \times 余因子_{11} + a_{12} \times 余因子_{12} + a_{13} \times 余因子_{13}$$

$$A = \begin{pmatrix} 1 & 4 & 0 \\ -5 & 7 & 1 \\ 3 & -2 & 0 \end{pmatrix}$$

の行列式の値を求めよ

余因子展開すると計算が容易

$$|A| = \begin{vmatrix} 1 & 4 & 0 \\ -5 & 7 & 1 \\ 3 & -2 & 0 \end{vmatrix}$$

$$= 0 \times (-1)^{1+1} \times \begin{vmatrix} -5 & 7 \\ 3 & -2 \end{vmatrix} + 1 \times (-1)^{2+3} \times \begin{vmatrix} 1 & 4 \\ 3 & -2 \end{vmatrix} + 0 \times (-1)^{3+3} \times \begin{vmatrix} 1 & 4 \\ -5 & 7 \end{vmatrix}$$

$$= 1 \times (-1) \times (-2 \times 1 - 3 \times 4) = 14$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \times 余因子_{11} + a_{12} \times 余因子_{12} + a_{13} \times 余因子_{13}$$

$$= a_{11} \times \tilde{a}_{11} + a_{12} \times \tilde{a}_{12} + a_{13} \times \tilde{a}_{13}$$

$$\tilde{a}_{11} = (-1)^{1+1} \times \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

余因子行列は、元の行列の転置となっている

$$\begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{21} & \tilde{a}_{31} \\ \tilde{a}_{12} & \tilde{a}_{22} & \tilde{a}_{32} \\ \tilde{a}_{13} & \tilde{a}_{23} & \tilde{a}_{33} \end{pmatrix} \xleftarrow{転置} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

元の行列に余因子行列をかけると

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{21} & \tilde{a}_{31} \\ \tilde{a}_{12} & \tilde{a}_{22} & \tilde{a}_{32} \\ \tilde{a}_{13} & \tilde{a}_{23} & \tilde{a}_{33} \end{pmatrix} = \begin{pmatrix} a_{11}\tilde{a}_{11} + a_{12}\tilde{a}_{12} + a_{13}\tilde{a}_{13} & & \\ a_{21}\tilde{a}_{11} + a_{22}\tilde{a}_{12} + a_{23}\tilde{a}_{13} & & \\ a_{31}\tilde{a}_{11} + a_{32}\tilde{a}_{12} + a_{33}\tilde{a}_{13} & & \end{pmatrix} = \begin{pmatrix} |A| & & \\ 0 & & \\ 0 & & \end{pmatrix}$$

$$a_{11}\tilde{a}_{11} + a_{12}\tilde{a}_{12} + a_{13}\tilde{a}_{13} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = |A|$$

$$a_{21}\tilde{a}_{11} + a_{22}\tilde{a}_{12} + a_{23}\tilde{a}_{13} = \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0$$

$$a_{31}\tilde{a}_{11} + a_{32}\tilde{a}_{12} + a_{33}\tilde{a}_{13} = \begin{vmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0$$

同じ行または列のある行列式の値は0になる

２及び3列目も同様に計算する。

→ 行列に余因子行列を掛けると対角成分が$|A|$で他が0となる行列になる

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{21} & \tilde{a}_{31} \\ \tilde{a}_{12} & \tilde{a}_{22} & \tilde{a}_{32} \\ \tilde{a}_{13} & \tilde{a}_{23} & \tilde{a}_{33} \end{pmatrix} = \begin{pmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{21} & \tilde{a}_{31} \\ \tilde{a}_{12} & \tilde{a}_{22} & \tilde{a}_{32} \\ \tilde{a}_{13} & \tilde{a}_{23} & \tilde{a}_{33} \end{pmatrix} = \begin{pmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{pmatrix}$$

$$\frac{1}{|A|} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{21} & \tilde{a}_{31} \\ \tilde{a}_{12} & \tilde{a}_{22} & \tilde{a}_{32} \\ \tilde{a}_{13} & \tilde{a}_{23} & \tilde{a}_{33} \end{pmatrix} = \frac{1}{|A|} \begin{pmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A^{-1} = \frac{1}{|A|} \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{21} & \tilde{a}_{31} \\ \tilde{a}_{12} & \tilde{a}_{22} & \tilde{a}_{32} \\ \tilde{a}_{13} & \tilde{a}_{23} & \tilde{a}_{33} \end{pmatrix}$$

余因子行列を行列式で割ると、逆行列が求まる

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = A$として$A^{-1}$逆行列を両辺にかける

$$A^{-1}A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A^{-1}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A^{-1}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$A^{-1} = \frac{1}{|A|}\begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{21} & \tilde{a}_{31} \\ \tilde{a}_{12} & \tilde{a}_{22} & \tilde{a}_{32} \\ \tilde{a}_{13} & \tilde{a}_{23} & \tilde{a}_{33} \end{pmatrix}$ ←余因子行列

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{|A|}\begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{21} & \tilde{a}_{31} \\ \tilde{a}_{12} & \tilde{a}_{22} & \tilde{a}_{32} \\ \tilde{a}_{13} & \tilde{a}_{23} & \tilde{a}_{33} \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \frac{1}{|A|}\begin{pmatrix} b_1\tilde{a}_{11} + b_2\tilde{a}_{21} + b_3\tilde{a}_{31} \\ b_1\tilde{a}_{12} + b_2\tilde{a}_{22} + b_3\tilde{a}_{32} \\ b_1\tilde{a}_{13} + b_2\tilde{a}_{23} + b_3\tilde{a}_{33} \end{pmatrix}$$

$$x_1 = \frac{b_1\tilde{a}_{11}+b_2\tilde{a}_{21}+b_3\tilde{a}_{31}}{|A|} \qquad x_2 = \frac{b_1\tilde{a}_{12}+b_2\tilde{a}_{22}+b_3\tilde{a}_{32}}{|A|} \qquad x_3 = \frac{b_1\tilde{a}_{13}+b_2\tilde{a}_{23}+b_3\tilde{a}_{33}}{|A|}$$

$$b_1\tilde{a}_{11} + b_2\tilde{a}_{21} + b_3\tilde{a}_{31} = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix} \qquad b_1\tilde{a}_{12} + b_2\tilde{a}_{22} + b_3\tilde{a}_{32} = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix} \qquad b_1\tilde{a}_{13} + b_2\tilde{a}_{23} + b_3\tilde{a}_{33} = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

$$a_{11} \times \tilde{a}_{11} + a_{21} \times \tilde{a}_{21} + a_{31} \times \tilde{a}_{31} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

クラメルの公式

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}} \qquad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}} \qquad x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}$$